A Characterization of First-Order Definable Subsets on Classes of Finite Total Orders

A. J. Kfoury* M. Wymann-Böni†

Boston University

August 1993

Abstract

We give an explicit and easy-to-verify characterization for subsets in finite total orders (infinitely many of them in general) to be uniformly definable by a first-order formula.

From this characterization we derive immediately that Beth's definability theorem does not hold in any class of finite total orders, as well as that McColm's first conjecture is true for all classes of finite total orders. Another consequence is a natural 0-1 law for definable subsets on finite total orders expressed as a statement about the possible densities of first-order definable subsets.

1 Introduction

Finite Model Theory has arisen as a complement to conventional model theory motivated by the search for models for databases and query languages. Also, the study of finite models can yield many beautiful characterizations of complexity classes in terms of logic. (For the first aspect, see the work of Aho and Ullman [1], Chandra [2], Chandra and Harel [3], Gaifman et.

*Partly supported by NSF grant CCR-9113196.

†Supported by a grant from the Swiss National Science Foundation.

al. [9], Immerman [12], Ioannides [14], Vardi [16]; for the second Fagin [7], Grädel [10], Immerman [13].)

Unfortunately, standard results and techniques from model theory like completeness, compactness and Beth's theorem cannot be applied to classes of finite models. In contrast, in finite model theory the combinatorial aspects of logic are crucial. One major issue in this subject is the question about the expressiveness of different logics on classes of finite structures which is connected to separation question for various complexity classes.

In the sequel, we isolate a property of first-order definable subsets, i.e., of unary first-order relations, and use it to improve some well-known results about the expressiveness of logics on finite structure to give more general results for this particular situation.

The notation follows the style of Chang and Keisler [4]. Structures are denoted by $\mathfrak{A}$, $\mathfrak{B}$, ... and we use the letters A, B, ... to denote the domains of $\mathfrak{A}$, $\mathfrak{B}$, ... For simplicity, we understand that whenever we mention a class, we mean a class containing an infinite number of mutually non-isomorphic structures. This is legitimate since in *finite* classes of finite structures no logic is more expressive than first-order logic, and this case is of no further interest.

2 Logical Games

As mentioned earlier, most techniques of general model theory fail in the context of finite models. There is however one technique which is still applicable: the use of Fraïssé-Ehrenfeucht games to describe logical equivalence of structures. (See Ehrenfeucht [6] and Fraïssé [8].)

Definition 2.1 *Given a finite language for a signature Σ without function symbols, we define the relation $\equiv_n$ for every $n \in \omega$, called n-equivalence, between structures for Σ by*

- $\mathfrak{A} \equiv_0 \mathfrak{B}$, *iff the submodels of $\mathfrak{A}$ and $\mathfrak{B}$ generated by the constants are isomorphic, or Σ has no constant symbols.*

- $\mathfrak{A} \equiv_{n+1} \mathfrak{B}$, *iff for every element $a \in A$ there is an element $b \in B$ such that $(\mathfrak{A}, a) \equiv_n (\mathfrak{B}, b)$, and for every $b \in B$ there is an $a \in A$ such that $(\mathfrak{A}, a) \equiv_n (\mathfrak{B}, b)$.*

These relations are best described as two-person games with perfect information played by two players, called the *spoiler* and the *duplicator*, on the two structures $\mathfrak{A}$ and $\mathfrak{B}$.

Every round of the game consists of a move by the spoiler and a response by the duplicator. In every round the spoiler has first the opportunity to choose the structure ($\mathfrak{A}$ or $\mathfrak{B}$) in which he wishes to play his move, then he marks an element of the structure. The response by the duplicator consists in choosing an element of the other structure ($\mathfrak{B}$ or $\mathfrak{A}$). In n rounds of this game the pairs of elements $(a_1, b_1), \ldots, (a_n, b_n)$ are accumulated. We say that the spoiler wins the n-move game on $\mathfrak{A}$ and $\mathfrak{B}$ iff the map given by

$$\begin{array}{lll} f: & c^{\mathfrak{A}} \mapsto c^{\mathfrak{B}} & \text{for every constant } c \text{ of } \Sigma \\ f: & a_i \mapsto b_i & \text{for } 1 \leq i \leq n \end{array}$$

is *not* a partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. If this map *is* a partial isomorphism then the duplicator wins the n-move game.

Hence, the objective of the spoiler in the game is to force the duplicator into breaking the partial isomorphism and thus exhibiting the difference between the structures $\mathfrak{A}$ and $\mathfrak{B}$. In contrast, the duplicator tries to preserve the partial isomorphism and to demonstrate the similarities of the two structures. He tries to duplicate every move by the spoiler by finding a response that is equivalent to the choice of the spoiler in the context of the game.

We say, that a player has a winning strategy for the n-move game on the structures $\mathfrak{A}$ and $\mathfrak{B}$ iff he has a method to win every n-move game.

It is not difficult to see that the previously defined relation $\equiv_n$ describes exactly the existence of a winning strategy for the duplicator for the n-move game.

We collect the prominent properties of the relations $\equiv_n$ without proof. For details and proofs, see Ehrenfeucht [6].

Theorem 2.2 (Ehrenfeucht)

- *For all n, the relation $\equiv_n$ is an equivalence relation.*
- *All relations $\equiv_n$ have finite index, i.e., there are finitely many first-order sentences $\chi_1, \ldots, \chi_{N(n)}$ with the following property: Every model satisfies exactly one of these sentences, and if two models $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentence then $\mathfrak{A} \equiv_n \mathfrak{B}$.*

- *If $\mathfrak{A} \equiv_n \mathfrak{B}$, then $\mathfrak{A}$ and $\mathfrak{B}$ satisfy exactly the same first-order sentences of quantifier-depth at most n.*
- *Two finite models $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic iff for all $n \in \omega$: $\mathfrak{A} \equiv_n \mathfrak{B}$.*

3 Characterization of First-Order Definable Subsets

All structures considered in this section are finite total orders. We assume a signature $\Sigma = \{\leq\}$ where $\leq$ is realized as a total order in every structure. Since all structures are finite, we deal with models for the theory of total discrete orders with endpoints. In this particularly simple situation it is safe to assume that all models are given as initial segments of the natural numbers. Thus, the unique model (up to isomorphism) of size m is $\mathfrak{A} = (\{0, \ldots, m-1\}, \leq)$.

As a first illustration of the use of games and for further reference we state a result about finite total orders:

Proposition 3.1 *For finite total orders $\mathfrak{A}$ and $\mathfrak{B}$: If $\|\mathfrak{A}\|, \|\mathfrak{B}\| > 2^n$, then $\mathfrak{A} \equiv_n \mathfrak{B}$.*

Proof:
By Theorem 2.2 the relation $\equiv_n$ is an equivalence, hence transitive, and we can assume that $\mathfrak{A} = [0, 2^n + l]$ and $\mathfrak{B} = [0, 2^n + l + 1]$. We have to show that the duplicator has a winning strategy for the n-move game on the structures $\mathfrak{A}$ and $\mathfrak{B}$. We use induction on the length of the game.

If $n = 0$, no moves are played. Since there are no constants in our context, the duplicator automatically wins regardless of the sizes of the models. (We follow the the convention that there are no empty structures.)

For $n = m+1$, assume that we have already established a winning strategy for all games up to m moves on all pairs of structures of size $> 2^m$. From this we have to show that there is a winning strategy for the duplicator in the $(m+1)$-move game on $\mathfrak{A}$ and $\mathfrak{B}$. This means that we have to find a response for the duplicator to the first move of the spoiler which guarantees that he can always successfully play in the remaining m-move game after the first pair of moves. If the first choice of the spoiler is $i \in A = [0, 2^n + l]$, the duplicator plays the same element $j = i$ in $B = [0, 2^n + l + 1]$ iff $i \leq \lfloor \|\mathfrak{A}\|/2 \rfloor$ is below the middle of $\mathfrak{A}$. Otherwise he plays the element $j = i + 1$ that has the same distance to the right endpoint of $\mathfrak{B}$ as i has from the endpoint of $\mathfrak{A}$. (Similarly, for a choice $i \in B$ by the spoiler, the duplicator uses $j = i$ as a response in the lower half and $j = i - 1$ for the upper half.)

Using this strategy the spoiler makes sure that of the two pairs of intervals $[0, i]$, $[0, j]$ and $[i, \|\mathfrak{A}\| - 1]$, $[j, \|\mathfrak{B}\| - 1]$ formed in the process, one is a pair of isomorphic intervals and the other pair consists of intervals which both are of a size larger than half the size of the smaller of the original structures, that is at least of size $> 2^m$. At this point, the duplicator can use the previously established strategy for the m-move game on each of the pairs of structures $[0, i]$, $[0, j]$ and $[i, \|\mathfrak{A}\| - 1]$, $[j, \|\mathfrak{B}\| - 1]$ separately. The latter is isomorphic to the pair of structures $[0, \|\mathfrak{A}\| - i - 1]$, $[0, \|\mathfrak{B}\| - j - 1]$, and the moves are to be shifted by the appropriate amounts.
□

Essentially the same argument is used to prove the following fact.

Lemma 3.2 *For finite total orders $\mathfrak{A}$ and $\mathfrak{B}$, and elements $a_1 < \ldots < a_l \in A$ and $b_1 < \ldots < b_l \in B$ the following holds:*

Given the intervals $I_0 = [0, a_1], I_j = [a_j, a_{j+1}]$ for $1 \leq j \leq l - 1$, and $I_l = [a_l, \|\mathfrak{A}\| - 1]$ in $\mathfrak{A}$ and analogous intervals J_j in $\mathfrak{B}$ (formed by the elements b_i):

If for all $j \in \{0, \ldots, l\}$, the intervals I_j and J_j are either isomorphic or longer than 2^n, then $(\mathfrak{A}, a_1, \ldots, a_l) \equiv_n (\mathfrak{B}, b_1, \ldots, b_l)$

We state a special case of this which we will use later.

Corollary 3.3 *For elements $a_1 < a_2$ in A: $(\mathfrak{A}, a_1) \equiv_n (\mathfrak{A}, a_2)$ if $2^n \leq a_1 < a_2 \leq \|\mathfrak{A}\| - 1 - 2^n$.*

Definition 3.4 *We call two elements $a_1, a_2 \in \mathfrak{A}$* n-indiscernible *in $\mathfrak{A}$ iff $(\mathfrak{A}, a_1) \equiv_n (\mathfrak{A}, a_2)$ and write "$a_1 \simeq_n a_2$ in $\mathfrak{A}$" for this relation.*

We drop the specification "in $\mathfrak{A}$" whenever possible without ambiguity.

Using Theorem 2.2 about games we can readily derive the following consequence.

Corollary 3.5 *For all unary formulae $\varphi(x)$ for the signature $\Sigma = \{\leq\}$ with quantifier-depth at most n:*

$$a_1 \simeq_n a_2 \qquad \textit{iff} \qquad a_1 \in \{a : \mathfrak{A} \models \varphi([a]\} \iff a_2 \in \{a : \mathfrak{A} \models \varphi([a]\}$$

Proof:
For all sentences φ with quantifier-depth at most n for the signature $\Sigma' = \{\leq, c\}$, where c is a new constant symbol, we have:

$$a_1 \simeq_n a_2 \qquad \text{iff} \qquad (\mathfrak{A}, a_1) \models \varphi(c) \iff (\mathfrak{A}, a_2) \models \varphi(c).$$

□

This means that all first-order definable unary relations respect the equivalence classes formed by n-indiscernible elements for all n larger than the quantifier-depth of the defining formula.

In particular, the situation for finite total orders is described by:

Proposition 3.6 *If $2^n \leq a_1 < a_2 \leq \|\mathfrak{A}\| - 1 - 2^n$, then $\mathfrak{A} \models \varphi[a_1] \iff \mathfrak{A} \models \varphi[a_2]$, for all φ of quantifier-depth at most n.*

Or, in other words: The interval $[2^n, \|\mathfrak{A}\| - 1 - 2^n]$ is either completely contained in the relation defined by $\varphi(x)$ or its complement (for n taken as the quantifier-depth of φ). Notice however that this behavior is independent of the size of the structure as long as it is large, i.e., this phenomenon is uniform with at most finitely many (small) structures as exceptions.

Thus, we have found a property which is particular to relations which are uniformly definable over the whole class by a first-order formula.

Now we turn to semantically given relations on a class of finite total orders.

Definition 3.7 *Call a family of subsets $\{R^{\mathfrak{A}}\}_{\mathfrak{A}\in\mathcal{C}}$ on a class $\mathcal{C}$ of finite total orders* strongly bounded *iff it satisfies the following condition:*

There is a number $N \in \omega$, called a bound for $\{R^{\mathfrak{A}}\}_{\mathfrak{A}\in\mathcal{C}}$, such that the interval $[N, \|\mathfrak{A}\| - N]$ is either totally inside $R^{\mathfrak{A}}$, or totally outside $R^{\mathfrak{A}}$, for all but finitely many models $\mathfrak{A} \in \mathcal{C}$.

We choose the notion *strongly bounded* because this condition states on one hand that the number of elements is uniformly bounded (or co-bounded) throughout the structures belonging to the class, and in addition, the elements cannot appear at arbitrary places. For example, the family $R^{\mathfrak{A}} = \{a : a = \lfloor \|\mathfrak{A}\|/2 \rfloor\}$ has a uniformly bounded number of elements, but it violates the condition stated in the definition.

Clearly, Proposition 3.6 states that families of subsets which are uniformly defined by a first-order formula φ, i.e., $R^{\mathfrak{A}} = \{a \in A : \mathfrak{A} \models \varphi[a]\}$ on all $\mathfrak{A} \in \mathcal{C}$, are strongly bounded.

In addition, the families of subsets defined by a first-order formula preserve membership for elements in the intervals $[0, N]$ and $[\|\mathfrak{A}\|-1-N, \|\mathfrak{A}\|-1]$ (near the endpoints) in all but finitely many models of the class (for one uniform N).

Lemma 3.8 *Given a unary first-order formula $\varphi(x)$, there is a constant M such that for all elements $a \in [0, M]$: either for almost all structures $\mathfrak{A} \in \mathcal{C}$, $\varphi[a]$ holds, or for almost all structures $\mathfrak{A} \in \mathcal{C}$, $\neg\varphi[a]$ holds. An analogous condition holds for the interval $[\|\mathfrak{A}\| - M - 1, \|\mathfrak{A}\| - 1]$. Thus, there is a uniform bound M, such that the interpretation of $\varphi(x)$ is the same near the endpoints in almost all models of the class.*

Proof:
There are formulae $\varphi_m(x)$ which define the m-th element of a structure. A simple choice could be:

$$\begin{aligned}\varphi_m(x) \equiv \quad (\exists x_1 \ldots x_{m-1})(\forall y)(\quad & x_1 < \ldots < x_{m-1} < x \\ & \wedge (y < x \rightarrow (y = x_1 \vee \ldots \vee y = x_{m-1})))\end{aligned}$$

Thus, it is possible to write a sentence $\hat{\varphi} \equiv \exists x(\varphi(x) \wedge \varphi_m(x))$ which is true exactly iff the m-th element is in the relation defined by $\varphi(x)$. By setting d to be the quantifier-depth of $\hat{\varphi}$, we get by Corollary 3.3 that this sentence is simultaneously true or false on all models of size $> 2^d$. The proof for elements near the top end of the model is the same.
□

Now we can conclude the characterization of first-order definable subsets on classes of finite total orders.

Theorem 3.9 *Let $\mathcal{C}$ be a class of finite total orders. A family of subsets $\{R^{\mathfrak{A}}\}_{\mathfrak{A}\in\mathcal{C}}$ is first-order definable iff it is strongly bounded for some bound N, and the patterns $R^{\mathfrak{A}}$ restricted to $[0, N]$ and restricted to $[\|\mathfrak{A}\|-1-N, \|\mathfrak{A}\|-1]$ respectively, are identical on all but finitely many models $\mathfrak{A} \in \mathcal{C}$.*

Proof:
We have already proved that all subsets defined by a first-order formula respect both properties. On the other hand it is simple to give a first-order definition for a family of subsets respecting both properties. Let N be the bound obtained from strong boundedness. Construct a formula

$$\begin{aligned}
\psi(x) \quad \equiv \quad & \bigvee\{x = n : n \in [0, N] \text{ and } n \in R^{\mathfrak{A}} \text{ for almost all } \mathfrak{A}\} \\
& \vee \ \bigvee\{\text{the same for the } N \text{ rightmost elements...}\} \\
& \vee \ (N < x < \|\mathfrak{A}\| - 1 - N) \\
& \vee \ \theta
\end{aligned}$$

(If the middle interval $[N, \|\mathfrak{A}\| - 1 - N]$ is omitted in the family, omit the third line of the formula.) The statements "$x = m$","$x = \|\mathfrak{A}\| - m$", and "$N < x < \|\mathfrak{A}\| - 1 - N$" are shorthand for the first-order formulae defining and using the m-th or m-th last elements. The formula θ realizes the finite list of exceptional structures. It is a disjunction composed of formulae $\xi_{\mathfrak{A}} \rightarrow (x = i_1 \vee \ldots \vee x = i_l)$, where $\xi_{\mathfrak{A}}$ is a sentence specifying the model $\mathfrak{A}$ up to isomorphism, and the conclusion of the implication is an explicit list of the elements to be included in the defined subset. The formula θ contains one such formula for every one of finitely many exceptions. This completes the proof.
$\square$

4 Applications of the Characterization

From our characterization we gain an improved refutation of well-known model theoretic theorems such as compactness and Beth's definability theorem. Gurevich showed in [11] that none of these (nor some other theorems which are true for the infinite case) hold in finite model theory. For this it suffices to exhibit a single class of structures violating the theorem. Our characterization of definable subsets allows us to strengthen this result in the special case of arbitrary infinite classes of finite structures for the vocabulary $\{\leq\}$, where this binary predicate is interpreted as linear order:

Theorem 4.1 *Beth's definability theorem fails on* every *infinite class of finite total orders.*

Recall that Beth's theorem states that implicit and explicit relations coincide in first-order logic (for infinite as well as finite models, but not for classes of finite models). Explicit relations are just the relations which can be defined by a first-order formula, whereas a family of implicit conditions of the form $\psi(x_1, \ldots, x_k; S)$, where S is a new relation symbol and ψ a first-order formula, is said to define a relation R implicitly, whenever the family of conditions has the *unique* solution R.
Proof:
By Theorem 3.9 the family of relations given by the even elements of every structure is not a first-order definable family of subsets, but it is easily defined using an implicit first-order definition:

$$S(0) \wedge (\forall x)(S(x) \rightarrow \neg S(x+1)) \wedge (\forall x)(\neg S(x) \rightarrow S(x+1)).$$

□

The difference between Gurevich's proof and ours, is that we know from the definability theorem that the relation which contains the even elements of every structure of the class is not definable in *any* infinite class of finite total orders, whereas he only establishes this for the classes of finite total orders that have models of even size, as well as models of odd size.

Another result is a confirmation of McColm's Conjecture for arbitrary classes of finite total orders. (For the details about this conjecture, see McColm [15].)

Theorem 4.2 *On every infinite class of finite total orders there is an inductive relation that is not definable by a first-order formula.*

Roughly speaking, an inductive (or fixed point) relation is generated by a first-order formula $\varphi(x_1, \ldots, x_k; S)$, where S is a special relation symbol occurring only in positive parts of φ. By setting $\varphi^0 = \emptyset$, $\varphi^{n+1} = \{(a_1, \ldots, a_k) : \varphi(a_1, \ldots, a_k; \varphi^n)\}$, and taking the limit $\varphi^\infty = \bigcup_n \varphi^n$, one can define the

least fixed point of the process which is the inductive relation defined by $\varphi(x_1, \ldots, x_k; S)$.
Proof:
The implicitly definable relation "a is even" is also definable using the least fixed point of φ below, but it is not elementary by Theorem 3.9.

$$\varphi(x; S) \equiv x = 0 \vee (\exists y)(S(y) \wedge x = y + 2)$$

□

As another interesting consequence, we obtain the following density law for first-order definable subsets:

Theorem 4.3 *On every infinite class of finite total orders, and for every unary first-order formula $\varphi(x)$ the limit*

$$\lim_{\mathfrak{A} \in \mathcal{C}} \frac{|\{a : \mathfrak{A} \models \varphi[a]\}|}{||\mathfrak{A}||}$$

exists and is either 0 or 1.

Proof:
Every first-order definable subset is strongly bounded by Theorem 3.9.
□

For more on 0-1 laws see Compton [5].

5 Concluding Remarks

The method we use is a simple game-theoretic approach. The applications shown above, are improved versions of well-known facts, in principle, and give sharper results in that they free the statements from explicit or implicit assumptions in the form: "Given *the* class $\mathcal{C}$ of *all* finite total orders ...", or describing some other specific class of finite total orders. The results show that it is safe to assume nothing further about the given class $\mathcal{C}$ than the natural fact that it contains infinitely many non-isomorphic structures (to eliminate the perfectly obvious case of a *finite* class of finite total orders). Also, the density theorem is a nice version of a strong 0-1 law which to our knowledge was never explicitly stated before.

It seems possible to generalize the present characterization of first-order definable subsets to obtain characterizations of first-order definable relations on finite total orders which are not necessarily unary.

The main direction for looking for new results would be an adaptation of this method to structures for more complicated languages than the one featuring order as its only relation. It seems however difficult to control the combinatorial complications in the presence of totally arbitrary relations in the language. Also the fact that these can be interpreted freely on every structure of the class without restrictions does not simplify the problem. One way to keep these complications at bay might be to first study classes which form a chain in the sense that smaller structures of the class are restrictions of larger structures to the smaller domain. This corresponds naturally to databases which can only be extended but not modified more generally.

References

[1] A. V. Aho and J. D. Ullman, *Universality of data retrieval languages*, Proc. 6th ACM Symp. on Principles of Programming Languages, 1979, pp. 110–117.

[2] A. Chandra, *Theory of database queries*, Proc. 7th ACM Symp. on Principles of Database Systems, 1988, pp. 1–9.

[3] A. Chandra and D. Harel, *Structure and complexity of relational queries*, Journal of Computer and System Sciences **25** (1982), 99–128.

[4] C. C. Chang and Keisler H. J., *Model theory*, 3rd ed., Studies in Logic, vol. 73, North-Holland, 1990.

[5] K. J. Compton, *0-1 laws in logic and combinatorics*, NATO Adv. Study Inst. on Algorithms and Order (I. Rival, ed.), Reidel, 1988, pp. 353–383.

[6] A. Ehrenfeucht, *An application of games to the completeness problem for formalized theories*, Fundamentæ Mathematicæ **49** (1961), 129–141.

[7] R. Fagin, *Monadic generalized spectra*, Zeitschrift für mathematische Logik und Grundlagen der Mathematik **21** (1975), 89–96.

[8] R. Fraïssé, *Sur quelques classifications des systèmes de relations*, Publ. Sci. Univ. Alger Ser. A **1** (1954), 35–185.

[9] H. Gaifman, H. Mairson, Y. Sagiv, and M. Y. Vardi, *Undecidable optimization problems for database logic programs*, Proc. 2nd IEEE Symp. on Logic in Computer Science, 1987, pp. 106–115.

[10] E. Grädel, *On logical descriptions of some concepts in structural complexity classes*, CSL '89: Proc. 3rd Workshop on Computer Science Logic (E. Börger, H. Kleine Buning, and M. M. Richter, eds.), Springer, 1990, pp. 163–175.

[11] Y. Gurevich, *Towards a logic tailored for computational complexity*, Computation and Proof Theory, LNCS, vol. 1104, Springer, 1984, pp. 175–216.

[12] N. Immerman, *Relational queries computable in polynomial time*, Information and Control **68** (1986), 76–98.

[13] ———, *Languages that capture complexity classes*, SIAM J. Comput. **16** (1987), 760–778.

[14] Y. E. Ioannides, *Bounded recursion in deductive databases*, Algorithmica **1** (1986), 86–104.

[15] G. L. McColm, *When is arithmetic possible?*, Ann. of Pure and Applied Logic **50** (1990), 29–51.

[16] M. Y. Vardi, *The complexity of relational query languages*, Proc. 14th ACM Symp. on Theory of Computing, 1982, pp. 137–146.